(1) $x+\sqrt{x^2-1}=t$ より $t-x=\sqrt{x^2-1}$

$$t^2-2tx+x^2=x^2-1$$

$$x=\frac{t^2+1}{2t} \cdots ①$$

また $\dfrac{dx}{dt}=\dfrac{t^2-1}{2t^2}$ より

$$dx=\frac{t^2-1}{2t^2}dx \cdots ②$$

よって ①. ② より

$$\int\sqrt{x^2-1}\,dx=\int(t-x)\cdot\frac{t^2-1}{2t^2}dt$$

$$=\int\left(t-\frac{t^2+1}{2t}\right)\cdot\frac{t^2+1}{2t^2}dt$$

$$=\frac{1}{4}\int\left(t-\frac{2}{t}+\frac{1}{t^3}\right)dt$$

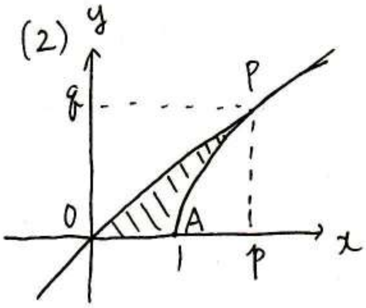
$$=\frac{1}{4}\left(\frac{1}{2}t^2-2\log|t|-\frac{1}{2t^2}\right)+C$$

$$\boxed{\frac{1}{t}=\frac{1}{x+\sqrt{x^2-1}}=x-\sqrt{x^2-1}}$$

$$=\frac{1}{8}\left\{\left(x+\sqrt{x^2-1}\right)^2-\left(x-\sqrt{x^2-1}\right)^2\right\}-\frac{1}{2}\log\left|x+\sqrt{x^2-1}\right|+C$$

$$=\underline{\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\log\left|x+\sqrt{x^2-1}\right|+C}$$

(2)



求める面積は図の斜線部分

$$S=\frac{1}{2}pq-\int_1^p\sqrt{x^2-1}\,dx$$

$$=\frac{1}{2}p\sqrt{p^2-1}-\left[\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\log\left|x+\sqrt{x^2-1}\right|\right]_1^p$$

$$=\frac{1}{2}p\sqrt{p^2-1}-\left(\frac{1}{2}p\sqrt{p^2-1}-\frac{1}{2}\log\left(p+\sqrt{p^2-1}\right)\right)$$

$$=\underline{\frac{1}{2}\log\left(p+\sqrt{p^2-1}\right)}$$

(3) $\dfrac{\theta}{2}=\dfrac{1}{2}\log\left(p+\sqrt{p^2-1}\right)$

$$\theta=\log\left(p+\sqrt{p^2-1}\right)$$

$$e^\theta=p+\sqrt{p^2-1}$$

$$(e^\theta-p)^2=p^2-1$$

よって $p=\dfrac{e^{2\theta}+1}{2e^\theta}=\underline{\dfrac{e^\theta+e^{-\theta}}{2}}$

$$q=\sqrt{p^2-1}=\underline{\frac{e^\theta-e^{-\theta}}{2}}$$