(1) $\displaystyle\int_{\frac{1}{n}}^{\frac{2}{n}} \frac{1}{x}\,dx = \Big[\log x\Big]_{\frac{1}{n}}^{\frac{2}{n}} = \log\frac{2}{n} - \log\frac{1}{n} = \underline{\log 2}$

$\displaystyle\int_{\frac{1}{n}}^{\frac{2}{n}} \frac{1}{\sin x}\,dx = \int_{\frac{1}{n}}^{\frac{2}{n}} \frac{\sin x}{1-\cos^2 x}\,dx$　　$\cos x = t$ とおくと $dt = -\sin x\,dx$

$\displaystyle = \int_{\alpha}^{\beta} \frac{-1}{1-t^2}\,dt$

| $x$ | $\frac{1}{n} \to \frac{2}{n}$ |
|---|---|
| $t$ | $\alpha \to \beta$ |

ただし $\alpha = \cos\frac{1}{n}$, $\beta = \cos\frac{2}{n}$

$\displaystyle = \frac{1}{2}\int_{\alpha}^{\beta}\left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt$

$\displaystyle = \frac{1}{2}\left[\log\left|\frac{t-1}{t+1}\right|\right]_{\alpha}^{\beta} = \frac{1}{2}\left[\log\frac{1-\cos x}{1+\cos x}\right]_{\frac{1}{n}}^{\frac{2}{n}}$

$\displaystyle = \frac{1}{2}\left[\log\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right]_{\frac{1}{n}}^{\frac{2}{n}} = \frac{1}{2}\left[\log\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)^2\right]_{\frac{1}{n}}^{\frac{2}{n}}$

$\displaystyle = \left[\log\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right]_{\frac{1}{n}}^{\frac{2}{n}} = \log\frac{\sin\frac{1}{n}}{\cos\frac{1}{n}} - \log\frac{\sin\frac{1}{2n}}{\cos\frac{1}{2n}}$

$\displaystyle = \underline{\log\frac{\sin\frac{1}{n}\cos\frac{1}{2n}}{\cos\frac{1}{n}\sin\frac{1}{2n}}}$　　または $\sin\frac{1}{n} = \sin 2\cdot\frac{1}{2n}$

$\displaystyle = 2\sin\frac{1}{2n}\cos\frac{1}{2n}$ として

$\displaystyle \log\frac{2\sin\frac{1}{2n}\cdot\cos^2\frac{1}{2n}}{\cos\frac{1}{n}\sin\frac{1}{2n}} = \log\frac{2\cos^2\frac{1}{2n}}{\cos\frac{1}{n}}$　としてもよい。

(2) $2\sin x < x + \sin x < 2x$ より、$\displaystyle\frac{1}{2x} < \frac{1}{x+\sin x} < \frac{1}{2\sin x}$

よって $\frac{1}{n}$ から $\frac{2}{n}$ までですべての辺を積分すると

$\displaystyle \frac{1}{2}\int_{\frac{1}{n}}^{\frac{2}{n}}\frac{dx}{x} < \int_{\frac{1}{n}}^{\frac{2}{n}}\frac{dx}{x+\sin x} < \frac{1}{2}\int_{\frac{1}{n}}^{\frac{2}{n}}\frac{dx}{\sin x}$　　$t = \frac{1}{2n}$ とおくと $n\to\infty$ のとき、$t \to 0$

(1)より $\displaystyle\int_{\frac{1}{n}}^{\frac{2}{n}}\frac{dx}{x} = \log 2$. $\displaystyle\int_{\frac{1}{n}}^{\frac{2}{n}}\frac{dx}{\sin x} = \log\frac{2\cos^2\frac{1}{2n}}{\cos\frac{1}{n}}$

また、$\displaystyle\lim_{n\to\infty}\log\frac{2\cos^2\frac{1}{2n}}{\cos\frac{1}{n}} = \lim_{t\to 0}\log\frac{2\cos^2 t}{\cos 2t} = \log 2$

よって、はさみうちの原理より、$\displaystyle\lim_{n\to\infty}\int_{\frac{1}{n}}^{\frac{2}{n}}\frac{dx}{x+\sin x} = \underline{\frac{1}{2}\log 2}$